EMA 3702

Strength of materials Lab rotary Some Technical Lecture Notes

Stress Terms

Stress is defined as force per unit area. It has the same units as pressure, and in fact pressure is one special variety of stress. However, stress is a much more complex quantity than pressure because it varies both with direction and with the surface it acts on.

Compression
: Stress that acts to shorten an object.

Tension
: Stress that acts to lengthen an object.

Normal Stress
: Stress that acts perpendicular to a surface. Can be either compressional or tensional.

Shear
: Stress that acts parallel to a surface. It can cause one object to slide over another. It also tends to deform originally rectangular objects into parallelograms. The most general definition is that shear acts to change the angles in an object.

Hydrostatic
: Stress (usually compressional) that is uniform in all directions. A scuba diver experiences hydrostatic stress. Stress in the earth is nearly hydrostatic. The term for uniform stress in the earth is **lithostatic**.

Directed Stress
: Stress that varies with direction. Stress under a stone slab is directed; there is a force in one direction but no counteracting forces perpendicular to it. This is why a person under a thick slab gets squashed but a scuba diver under the same pressure doesn't. The scuba diver feels the same force in all directions.

We only see the results of stress as it deforms materials. Even if we were to use a strain gauge to measure in-situ stress in the materials, we would not measure the stress itself. We would measure the deformation of the strain gauge (that's why it's called a "*strain* gauge") and use that to infer the stress.

Strain Terms

Strain is defined as the amount of deformation an object experiences compared to its original size and shape. For example, if a block 10 cm on a side is deformed so that it becomes 9 cm long, the strain is (10-9)/10 or 0.1 (sometimes expressed in percent, in this case 10 percent.) Note that strain is dimensionless.

$$\varepsilon = (\delta L)/L$$

Longitudinal or Linear Strain

Strain that changes the length of a line without changing its direction. Can be either compressional or tensional.

Compression

Longitudinal strain that shortens an object.

Tension

Longitudinal strain that lengthens an object.

Shear

Strain that changes the angles of an object. Shear causes lines to rotate.

Infinitesimal Strain

Strain that is tiny, a few percent or less. Allows a number of useful mathematical simplifications and approximations.

Finite Strain

Strain larger than a few percent. Requires a more complicated mathematical treatment than infinitesimal strain.

Homogeneous Strain

Uniform strain. Straight lines in the original object remain straight. Parallel lines remain parallel. Circles deform to ellipses. Note that this definition rules out folding, since an originally straight layer has to remain straight.

Inhomogeneous Strain

How real geology behaves. Deformation varies from place to place. Lines may bend and do not necessarily remain parallel.

Terms for Behavior of Materials

Elastic

Material deforms under stress but returns to its original size and shape when the stress is released. There is no permanent deformation. Some elastic strain, like in a rubber band, can be large, but in rocks it is usually small enough to be considered infinitesimal.

Brittle

Material deforms by fracturing. Glass is brittle. Rocks are typically brittle at low temperatures and pressures.

Ductile

Material deforms without breaking. Metals are ductile. Many materials show both types of behavior. They may deform in a ductile manner if deformed slowly, but fracture if deformed too quickly or too much. Rocks are typically ductile at high temperatures or pressures.

Viscous

Materials that deform steadily under stress. Purely viscous materials like liquids deform under even the smallest stress. Rocks may behave like viscous materials under high temperature and pressure.

Plastic

Material does not flow until a threshold stress has been exceeded.

Viscoelastic

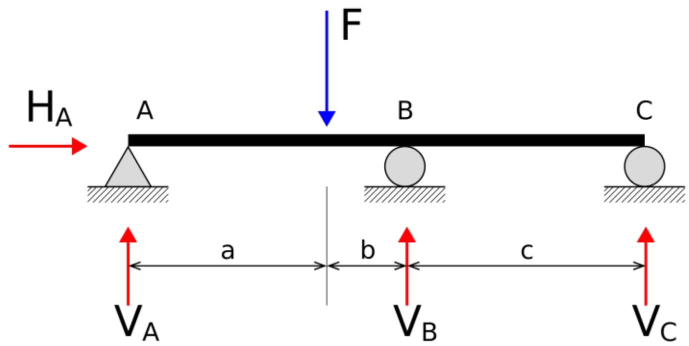
Combines elastic and viscous behavior. Models of glacio-isostasy frequently assume a viscoelastic earth: the crust flexes elastically and the underlying mantle flows viscously.

Beams

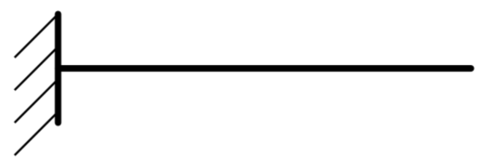
A **beam** is a structural member which carries loads. These loads are most often perpendicular to its longitudinal axis, but they can be of any geometry. A beam supporting any load develops internal stresses to resist applied loads. These internal stresses are bending stresses, shearing stresses, and normal stresses.

Beam types are determined by method of support, not by method of loading. Below are three types of beams that will be investigated in this course:

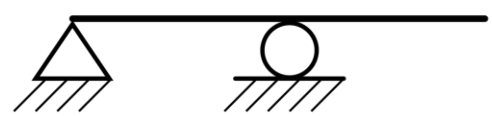
1. Simple Support Beam:



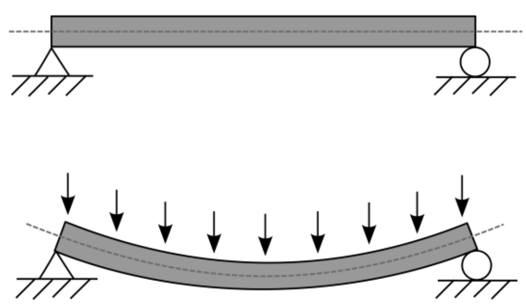
2. Cantilever Beam:



3. **Indeterminate Statically**



Beam Support



The first two types are statically determinate, meaning that the reactions, shears and moments can be found by the laws of statics alone. Continuous beams are statically indeterminate. The internal forces of these beams cannot be found using the laws of statics alone. Early structures were designed to be statically determinate because simple analytical methods for the accurate structural analysis of indeterminate structures were not developed until the first part of this century. A number of formulas have been derived to simplify analysis of indeterminate beams.

Beam Loading Conditions:

The two beam loading conditions that either occur separately, or in some combination, are:

A. Concentrated Load
B. Distributed Laod

CONCENTRATED
Either a force or a moment can be applied as a concentrated load. Both are applied at a single point along the axis of a beam. These loads are shown as a "jump" in the shear or moment diagrams. The point of application for such a load is indicated in the diagram above. Note that this is NOT a hinge! It is a point of application. This could be point at which a railing is attached to a bridge, or a lamppost on the same.

DISTRIBUTED
Distributed loads can be uniformly or non-uniformly distributed. Both types are commonly found on all kinds of structures. Distributed loads are shown as an angle or curve in the shear or moment diagram. A uniformly distributed load can evolve into a one with unevenly uniformly distributed load (snow melting to ice at the edge of a roof), but are normally assumed to act as given. These loads are often replaced by a singular resultant force in order to simplify the structural analysis.

Introduction Beam Design:

Normally a beam is analyzed to obtain the maximum stress and this is compared to the material strength to determine the design safety margin. It is also normally required to calculate the deflection on the beam under the maximum expected load. The determination of the maximum stress results from producing the shear and bending moment diagrams. To facilitate this work the first stage is normally to determine all of the external loads.

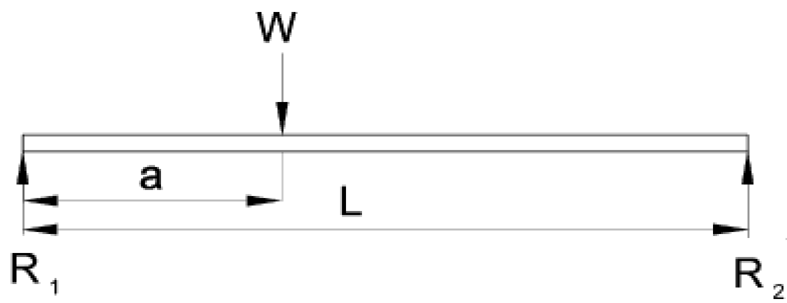
Nomenclature

e = strain
σ = stress (N/m^2)
E = Young's Modulus = σ /e (N/m^2)
y = distance of surface from neutral surface (m).
R = Radius of neutral axis (m).
I = Moment of Inertia (m^4 - more normally cm^4)
Z = section modulus = I/y_{max} (m^3 - more normally cm^3)
M = Moment (Nm)
w = Distributed load on beam (kg/m) or (N/m as force units)

W = total load on beam (kg) or (N as force units)
F= Concentrated force on beam (N)
S= Shear Force on Section (N)
L = length of beam (m)
x = distance along beam (m)

Calculation of external forces

To allow determination of all of the external loads a free-body diagram is construction with all of the loads and supports replaced by their equivalent forces. A typical free-body diagram is shown below.



The unknown forces (generally the support reactions) are then determined using the equations for plane static equilibrium.

Equations for plane static equilibrium (in two dimension)

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

For example considering the simple beam above the reaction R_2 is determined by Summing the moments about R_1 to zero

$$R_2 . L - W.a = 0 \text{ Therefore } R_2 = W.a / L$$

R_1 is determined by summing the vertical forces to 0

$$W - R_1 - R_2 = 0 \text{ Therefore } R_1 = W - R_2$$

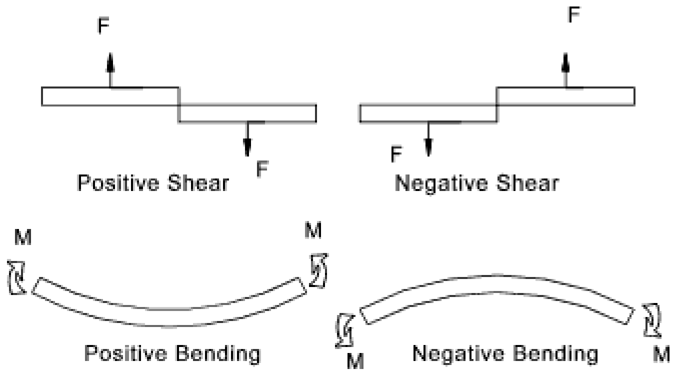
Shear and Bending Moment Diagram

The shear force diagram indicates the shear force withstood by the beam section along the length of the beam.
The bending moment diagram indicates the bending moment withstood by the beam section along the length of the beam.
It is normal practice to produce a free body diagram with the shear diagram and the bending moment diagram position below

For simply supported beams the reactions are generally simple forces. When the beam is built-in the free body diagram will show the relevant support point as a reaction force and a reaction moment....

Sign Convention

The sign convention used for shear force diagrams and bending moments is only important in that it should be used consistently throughout a project. The sign convention used on this page is as below.

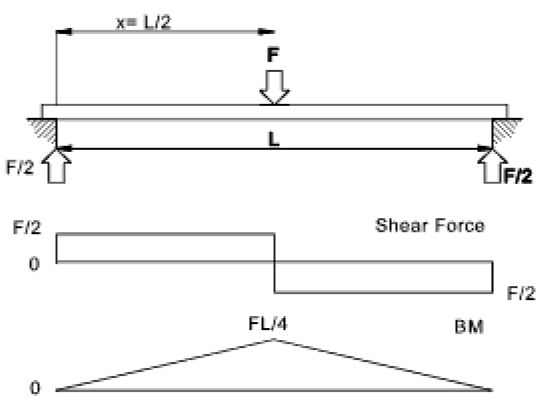


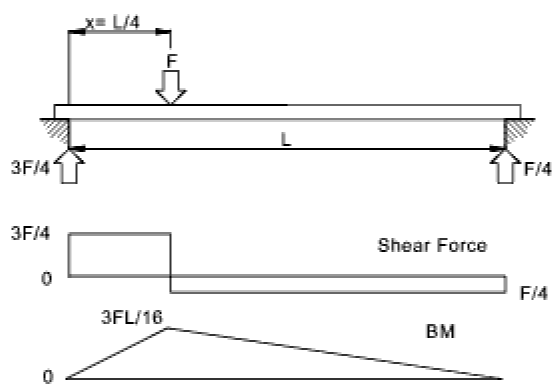
Typical Diagrams

A shear force diagram is simply constructed by moving a section along the beam from (say) the left origin and summing the forces to the left of the section. The equilibrium condition states that the forces on either side of a section balance and therefore the resisting shear force of the section is obtained by this simple operation

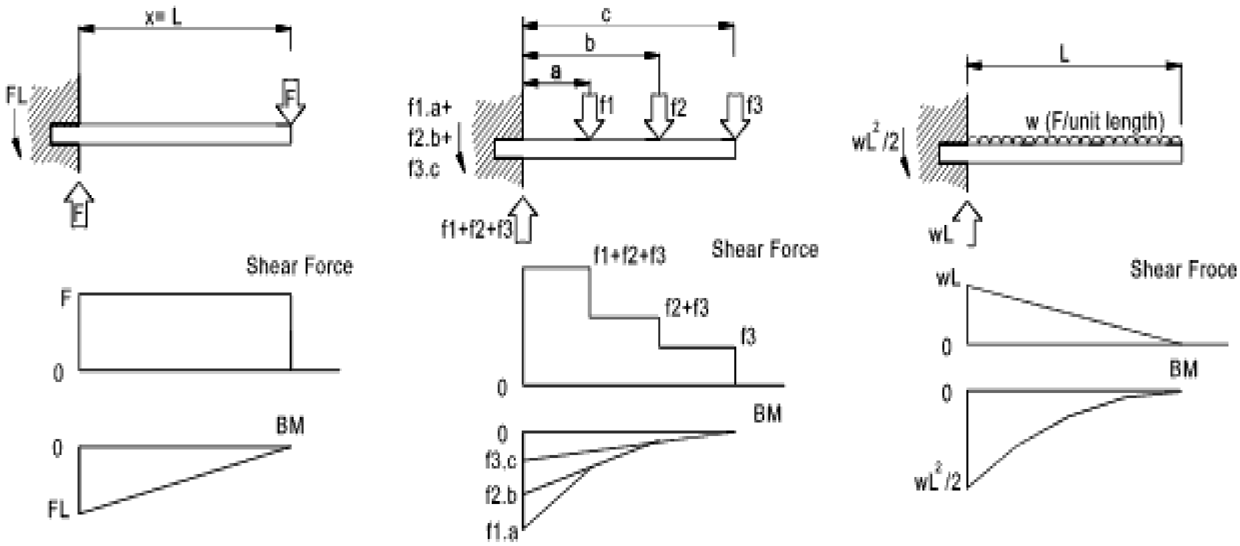
The bending moment diagram is obtained in the same way except that the moment is the sum of the product of each force and its distance(x) from the section. Distributed loads are calculated buy summing the product of the total force (to the left of the section) and the distance(x) of the centroid of the distributed load.

The sketches below show simply supported beams with on concentrated force.





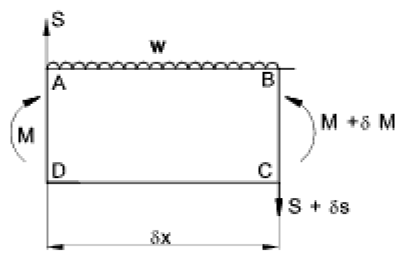
The sketches below show Cantilever beams with three different load combinations.



Note: The force shown if based on loads (weights) would need to be converted to force units i.e. 50kg = 50x9,81(g) = 490 N.

Shear Force Moment Relationship

Consider a short length of a beam under a distributed load separated by a distance δx.



The bending moment at section AD is M and the shear force is S. The bending moment at BC = M + δM and the shear force is S + δS.

The equations for equilibrium in 2 dimensions results in the equations.. Forces.

$$S - w.\delta x = S + \delta S$$

Therefore making δx infinitely small then.. $dS/dx = -w$

Moments.. Taking moments about C

$$M + S\delta x - M - \delta M - w(\delta x)^2/2 = 0$$

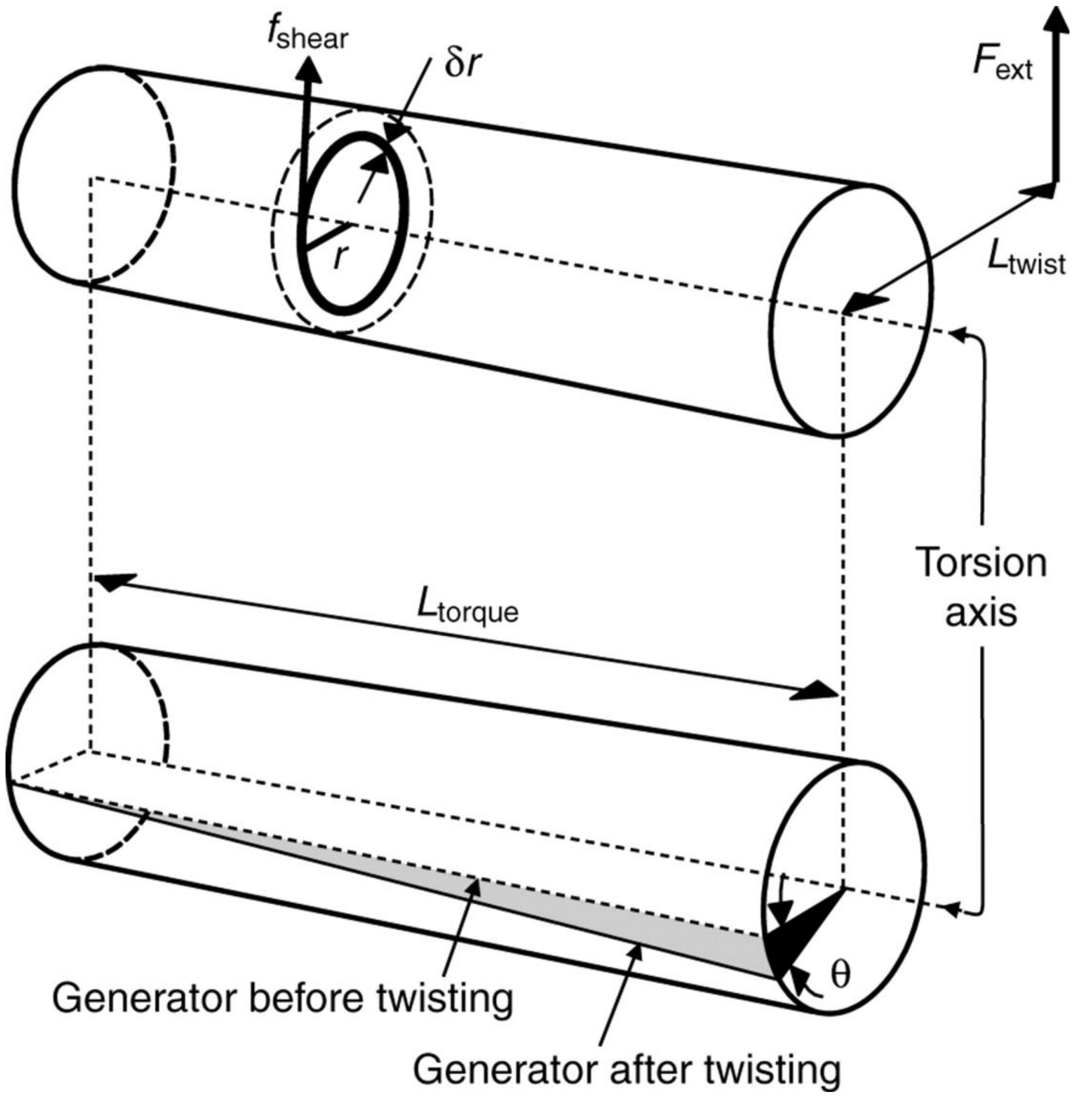
Therefore making δx infinitely small then.. $dM/dx = S$

Therefore putting the relationships into integral form.

$$-\int w.dx = \int dS = S \quad ..and.. \quad \int S.dx = \int dM = M$$

The integral (Area) of the shear diagram between any limits results in the change of the shearing force between these limits and the integral of the Shear Force diagram between limits results in the change in bending moment...

Torsion (mechanics)



In solid mechanics, **torsion** is the twisting of an object due to an applied torque. In circular sections, the resultant shearing stress is perpendicular to the radius.

For solid or hollow shafts of uniform circular cross-section and constant wall thickness, the torsion relations are:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\varphi}{\ell}$$

where:

- R is the outer radius of the shaft.
- τ is the maximum shear stress at the outer surface.

- φ is the angle of twist in radians.
- T is the torque (N·m or ft·lbf).
- ℓ is the length of the object the torque is being applied to or over.
- G is the shear modulus or more commonly the modulus of rigidity and is usually given in gigapascals (GPa), lbf/in^2 (psi), or lbf/ft^2.
- J is the torsion constant for the section . It is identical to the polar moment of inertia for a round shaft or concentric tube only. For other shapes J must be determined by other means. For solid shafts the membrane analogy is useful, and for thin walled tubes of arbitrary shape the shear flow approximation is fairly good, if the section is not re-entrant. For thick walled tubes of arbitrary shape there is no simple solution, and FEA may be the best method.
- the product GJ is called the torsional rigidity.

The shear stress at a point within a shaft is:

$$\tau_{\varphi_z} = \frac{Tr}{J}$$

where:

- r is the distance from the center of rotation

Note that the highest shear stress is at the point where the radius is maximum, the surface of the shaft. High stresses at the surface may be compounded by stress concentrations such as rough spots. Thus, shafts for use in high torsion are polished to a fine surface finish to reduce the maximum stress in the shaft and increase its service life.

The angle of twist can be found by using:

$$\varphi = \frac{T\ell}{JG}.$$

Polar moment of inertia

The polar moment of inertia for a solid shaft is:

$$J = \frac{\pi}{2} r^4$$

where r is the radius of the object.

The polar moment of inertia for a pipe is:

$$J = \frac{\pi}{2}(r_o^4 - r_i^4)$$

where the o and i subscripts stand for the outer and inner radius of the pipe.

For a thin cylinder

$$J = 2\pi R^3 t$$

where R is the average of the outer and inner radius and t is the wall thickness.

Failure mode

The shear stress in the shaft may be resolved into principal stresses via Mohr's circle. If the shaft is loaded only in torsion then one of the principal stresses will be in tension and the other in compression. These stresses are oriented at a 45 degree helical angle around the shaft. If the shaft is made of brittle material then the shaft will fail by a crack initiating at the surface and propagating through to the core of the shaft fracturing in a 45 degree angle helical shape. This is often demonstrated by twisting a piece of blackboard chalk between one's fingers.

Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.

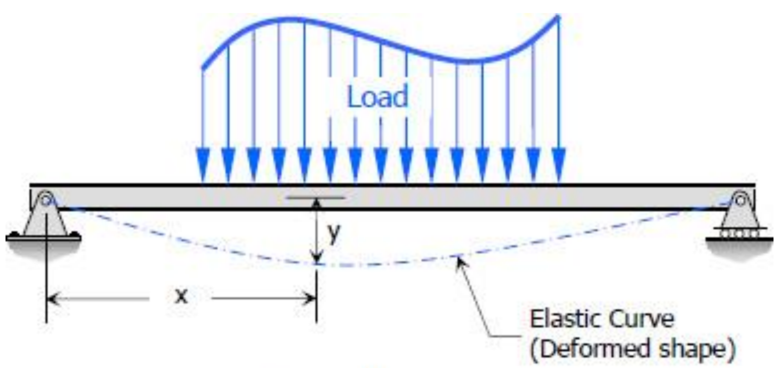


Figure: Elastic curve

Methods of Determining Beam Deflections

Numerous methods are available for the determination of beam deflections. These methods include:

1. Double-integration method
2. Area-moment method
3. Strain-energy method (Castigliano's Theorem)

4. Three-moment equation
5. Conjugate-beam method
6. Method of superposition
7. Virtual work method

Of these methods, the first two are the ones that are commonly used.

Introduction

The stress, strain, dimension, curvature, elasticity, are all related, under certain assumption, by the theory of simple bending. This theory relates to beam flexure resulting from couples applied to the beam without consideration of the shearing forces.

Superposition Principle

The superposition principle is one of the most important tools for solving beam loading problems allowing simplification of very complicated design problems..

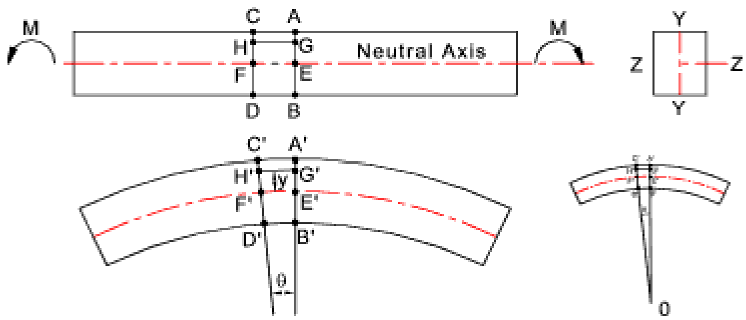
For beams subjected to several loads of different types the resulting shear force, bending moment, slope and deflection can be found at any location by summing the effects due to each load acting separately to the other loads.

Nomenclature

e = strain
E = Young's Modulus = σ /e (N/m^2)
y = distance of surface from neutral surface (m).
R = Radius of neutral axis (m).
I = Moment of Inertia (m^4 - more normally cm^4)
Z = section modulus = I/y_{max}(m^3 - more normally cm^3)
F = Force (N)
x = Distance along beam
δ = deflection (m)
θ = Slope (radians)
σ = stress (N/m^2)

Simple Bending

A straight bar of homogeneous material is subject to only a moment at one end and an equal and opposite moment at the other end...



Assumptions

The beam is symmetrical about Y-Y
The traverse plane sections remain plane and normal to the longitudinal fibres after bending (Beroulli's assumption)
The fixed relationship between stress and strain (Young's Modulus)for the beam material is the same for tension and compression ($\sigma = E.e$)

Consider two section very close together (AB and CD).
After bending the sections will be at A'B' and C'D' and are no longer parallel. AC will have extended to A'C' and BD will have compressed to B'D'
The line EF will be located such that it will not change in length. This surface is called neutral surface and its intersection with Z_Z is called the neutral axis
The development lines of A'B' and C'D' intersect at a point 0 at an angle of θ radians and the radius of E'F' = R
Let y be the distance(E'G') of any layer H'G' originally parallel to EF..Then

$$H'G'/E'F' = (R+y)\theta / R\theta = (R+y)/R$$

And the strain e at layer H'G' =

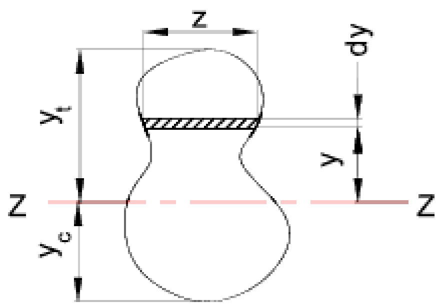
$$e = (H'G' - HG) / HG = (H'G' - HG) / EF = [(R+y)\theta - R\theta] / R\theta = y/R$$

The accepted relationship between stress and strain is $\sigma = E.e$ Therefore

$$\sigma = E.e = E.\ y/R$$
$$\sigma / E = y / R$$

Therefore, for the illustrated example, the tensile stress is directly related to the distance above the neutral axis. The compressive stress is also directly related to the distance below the neutral axis. Assuming E is the same for compression and tension the relationship is the same.

As the beam is in static equilibrium and is only subject to moments (no vertical shear forces) the forces across the section (AB) are entirely longitudinal and the total compressive forces must balance the total tensile forces. The internal couple resulting from the sum of (σ.dA .y) over the whole section must equal the externally applied moment.



$\sum(\sigma.\delta A) = 0$ therefore $\sum(\sigma.z\ \delta y) = 0$

As $\sigma = \frac{yE}{R}$ therefore $\frac{E}{R}\sum(y.\ \delta A) = 0$ and $\frac{E}{R}\sum(y.\ z\delta y) = 0$

This can only be correct if Σ(yδa) or Σ(y.z.δy) is the moment of area of the section about the neutral axis. This can only be zero if the axis passes through the centre of gravity (centroid) of the section.

The internal couple resulting from the sum of (σ.dA .y) over the whole section must equal the externally applied moment. Therefore the couple of the force resulting from the stress on each area when totalled over the whole area will equal the applied moment

The force on each area element = $\sigma.\ \delta A = \sigma.\ z.\ \delta y$

The resulting moment = $y.\sigma.\ \delta A = \sigma.\ z.\ y.\ \delta y$

The total moment $M = \sum(y.\sigma.\delta A)$ and $\sum(\sigma.z.y\ \delta y)$

Using $\frac{E}{R}y = \sigma$

$M = \frac{E}{R}\sum(y^2.\ \delta A)$ and $\frac{E}{R}\sum(z.y^2\delta y)$

$\sum(y^2.\delta A)$ is the Moment of Inertia of the section(I)

From the above the following important simple beam bending relationship results

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$$

It is clear from above that a simple beam subject to bending generates a maximum stress at the surface furthest away from the neutral axis. For sections symmetrical about Z-Z the maximum compressive and tensile stress is equal.

$$\sigma_{max} = y_{max}.\ M / I$$

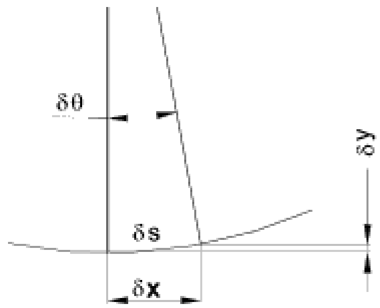
The factor I / y_{max} is given the name section Modulus (Z) and therefore

$$\sigma_{max} = M / Z$$

Values of Z are provided in the tables showing the properties of standard steel sections

Deflection of Beams

Below is shown the arc of the neutral axis of a beam subject to bending.



For small angle dy/dx = tan θ = θ
The curvature of a beam is identified as dθ /ds = 1/R
In the figure δθ is small and δx; is practically = δs; i.e ds /dx =1

$$\frac{1}{R} = \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

From this simple approximation the following relationships are derived.

$$\frac{1}{R} = \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\text{Slope} = \theta = \frac{dy}{dx} = \int\left(\frac{d^2y}{dx^2}\right)dx = \int \frac{M}{EI}\,dx$$

Integrating between selected limits.

The deflection between limits is obtained by further integration.

$$\text{Deflection} = x = \int \theta\,dx = \int\left(\frac{dy}{dx}\right)dx = \int\int \frac{M}{EI}\,dx$$

It has been proved ref Shear - Bending that dM/dx = S and dS/dx = -w = d^2M /dx
Where S = the shear force M is the moment and w is the distributed load /unit length of beam. therefore

$$S = \frac{dy}{dx}\left(EI\frac{d^2y}{dx^2}\right) = EI\frac{d^3y}{dx^3} \text{ and } -w = EI\frac{d^4y}{dx^4}$$

If w is constant or a integratatable function of x then this relationship can be used to arrive at general expressions for S, M, dy/dx, or y by progressive integrations with a constant of integration being added at each stage. The properties of the supports or fixings may be used to determine the constants. (x= 0 - simply supported, dx/dy = 0 fixed end etc)

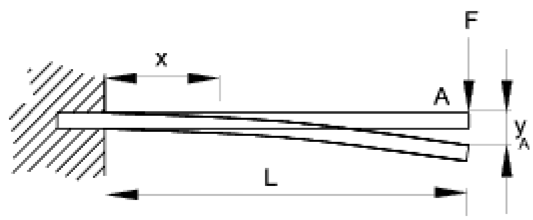
In a similar manner if an expression for the bending moment is known then the slope and deflection can be obtained at any point x by single and double integration of the relationship and applying suitable constants of integration.

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Singularity functions can be used for determining the values when the loading a not simple ref Singularity Functions

Example - Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0



From the equilibrium balance ..At the support there is a resisting moment -FL and a vertical upward force F.
At any point x along the beam there is a moment $F(x - L) = M_x = EI\, d^2y/dx^2$

$$EI\frac{d^2y}{dx^2} = -F(L-x) \quad \text{Integrating}$$

$$EI\frac{dy}{dx} = -F\left(Lx - \frac{x^2}{2}\right) + C_1 \quad .....(C_1\text{=0 because dy/dx = 0 at x = 0 })$$

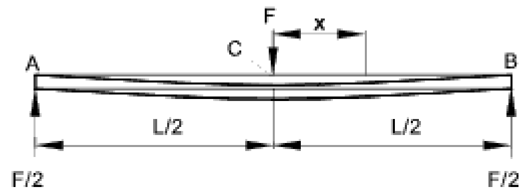
Integrating again

$$EI\,y = -F\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2 \quad .....(C\text{ =0 because y = 0 at x = 0 })$$

$$\text{At end A} \quad \left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}\left(L^2 - \frac{L^2}{2}\right) = -\frac{FL^2}{2EI} \quad \text{and} \quad y_A = -\frac{F}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{FL^3}{3EI}$$

Example - Simply supported beam

Consider a simply supported uniform section beam with a single load F at the centre. The beam will be deflect symmetrically about the centre line with 0 slope (dy/dx) at the centre line. It is convenient to select the origin at the centre line.



$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{F}{2}\left(\frac{L}{2}+x\right) - Fx\right] = \frac{F}{2EI}\left(\frac{L}{2}-x\right) \quad \text{Integrating}$$

$$\frac{dy}{dx} = \frac{F}{2EI}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) + C_1 \ldots \; (C_1 = 0 \text{ because } dy/dx = 0 \text{ at } x = 0)$$

$$\text{Integrating again } y = \frac{F}{2EI}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) + C_2$$

$$y = 0 \text{ when } x = L/2 \text{ therefore } \frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) + c_2 = 0$$

$$\text{and thus } c_2 = -\frac{FL^3}{48EI}$$

$$\text{At end B} \quad \left(\frac{dy}{dx}\right)_b = -\frac{F}{2EI}\left(\frac{L^2}{4} - \frac{L^2}{8}\right) = \frac{FL^2}{16EI} \quad \text{and} \quad y_B = \frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) - \frac{FL^3}{48EI} = 0$$

At centre C
x = 0

$$y_c = -\frac{Fl^3}{48EI} \qquad \left(\text{slope } \frac{dy}{dx} = 0 \text{ by symmetry}\right)$$

Moment Area Method

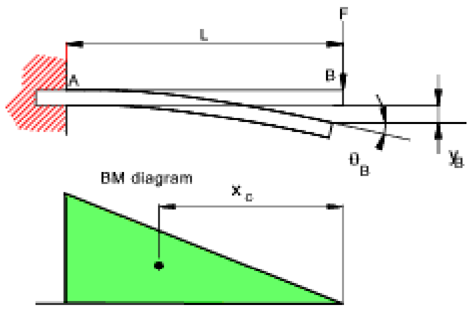
This is a method of determining the change in slope or the deflection between two points on a beam. It is expressed as two theorems...

Theorem 1
If A and B are two points on a beam the change in angle (radians) between the tangent at A and the tangent at B is equal to the area of the bending moment diagram between the points divided by the relevant value of EI (the flexural rigidity constant).

Theorem 2
If A and B are two points on a beam the displacement of B relative to the tangent of the beam at A is equal to the moment of the area of the bending moment diagram between A and B about the ordinate through B divided by the relevant value of EI (the flexural rigidity constant).

Examples ..Two simple examples are provide below to illustrate these theorems

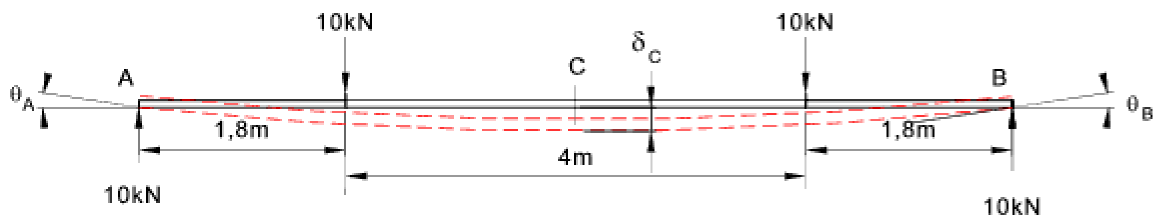
Example 1) Determine the deflection and slope of a cantilever as shown..

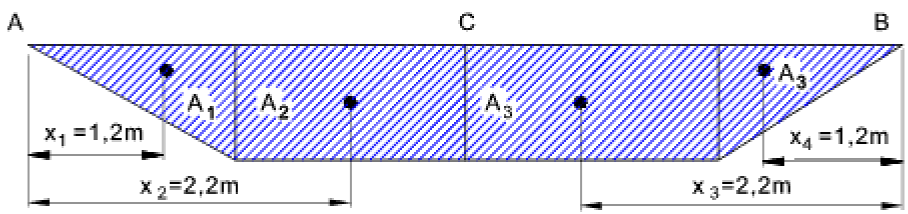


The bending moment at A = M_A = FL
The area of the bending moment diagram $A_M = F.L^2/2$
The distance to the centroid of the BM diagram from B= $x_c = 2L/3$
The deflection of B = $y_b = A_M.x_c/EI = F.L^3/3EI$
The slope at B relative to the tan at A = $\theta_b = A_M/EI = FL^2/2EI$

Example 2) Determine the central deflection and end slopes of the simply supported beam as shown..

E = 210 GPa ......I = 834 cm^4...... EI = 1,7514. 10 $^6Nm^2$





Bending Moment Diagram

A_1 = 10.1,8.1,8/2 = 16,2kNm
A_2 = 10.1,8.2 = 36kNm
A_2 = 10.1,8.2 = 36kNm
A_1 = 10.1,8.1,8/2 = 16,2kNm
x_1 = Centroid of A_1 = (2/3).1,8 = 1,2
x_2 = Centroid of A_2 = 1,8 + 1 = 2,8
x_3 = Centroid of A_3 = 1,8 + 1 = 2,8
x_4 = Centroid of A_4 = (2/3).1,8 = 1,2

The slope at A is given by the area of the moment diagram between A and C divided by EI.

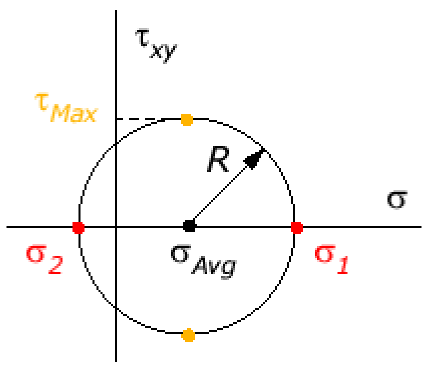
$$\theta_A = (A_1 + A_2)/EI = (16{,}2+36).10^{3} / (1{,}7514.\ 10^{6})$$
$$= 0{,}029\text{rads} = 1{,}7 \text{ degrees}$$

The deflection at the centre (C) is equal to the deviation of the point A above a line that is tangent to C.
Moments must therefore be taken about the deviation line at A.

$$\delta_C = (A_M.x_M)/EI = (A_1 x_1 + A_2 x_2) / EI = 120{,}24.10^{3} / (1{,}7514.\ 10^{6})$$
$$= 0{,}0686\text{m} = 68{,}6\text{mm}$$

Mohr's Circle

Introduced by Otto Mohr in 1882, Mohr's Circle illustrates principal stresses and stress transformations via a graphical format,



The two principal stresses are shown in **red**, and the maximum shear stress is shown in **orange**. Recall that the normal stresses equal the principal stresses when the stress element is aligned with the principal directions, and the shear stress equals the maximum shear stress when the stress element is rotated 45° away from the principal directions.

As the stress element is rotated away from the principal (or maximum shear) directions, the normal and shear stress components will always lie on Mohr's Circle.

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

Derivation of Mohr's Circle

To establish Mohr's Circle, we first recall the stress transformation formulas for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\square + \sin^2 2\square = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret $\square_x$ and $\square_y$ as being the two principal stresses, and $\square_{xy}$ as being the maximum shear stress. Then we can define the average stress, $\square_{avg}$, and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{\text{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{\text{Avg}}\right)^2 + \tau_{x'y'}{}^2 = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,